Chebyshev Polynomial Fit for Terrain Elevation

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19961220 082

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Adelphi, MD 20783-1197

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Abstract

There is currently a desire to use Chebyshev polynomials to fit terrain elevation data. Such a fit would create a surface function that exactly fits the known elevations, and would describe an elevation at any point on that surface. This note questions the appropriateness of using Chebyshev polynomials for this purpose, as opposed to linear interpolation or use of a cubic spline. A set of elevations in one direction is used to illustrate a point that large transitions in elevation influence the coefficients in the polynomial fit and contribute spectral energy to points far from the transition area. It argues that a linear interpolation process, or a cubic spline interpolation, would be more appropriate.

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1. Introduction

There is a concern whether a Chebyshev polynomial curve fit to elevation data is an appropriate method for interpolating the elevation values for points between those that are known, such as those given by the Defense Mapping Agency Digital Terrain Elevation Data, level 1 coverage. In brief, a Chebyshev polynomial is defined as (Hamming, 1986; Press et al, 1994)

$$T_n(x) = \cos(n \cos^{-1}(x))$$

or

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$

$$\ldots$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad n \geq 2 \ .$$

A curve fit using Chebyshev polynomials takes on the form

$$f(x) = \left[\sum_{k=1}^{N} c_k T_{k-1}(x) \right] - \frac{1}{2} c_1 \ ,$$

where the Chebyshev polynomial coefficients, c_k, are determined from

$$c_k = \frac{2}{N} \sum_{n=1}^{N} f(x_n) T_{k-1}(x_n) \ ,$$

where

N = total number of data points to fit,

$f(x)$ = exact function described by a Chebyshev polynomial curve fit,

$f(x_n)$ = data point at location n, and

c_k = Chebyshev polynomial coefficient for polynomial $k - 1$.

By taking advantage of the polynomial's orthogonality, we can determine the coefficients in a simpler manner by substituting for x_n with

$$x = \cos\left(\frac{\pi(k - \frac{1}{2})}{n} \right) \ ,$$

which results in

$$c_k = \frac{2}{N} \sum_{n=1}^{N} f\left[\cos\left(\frac{\pi(n - \frac{1}{2})}{N} \right) \right] \cos\left(\frac{\pi(k - 1)(n - \frac{1}{2})}{N} \right) \ .$$

This last relationship is important because it gives the Chebyshev polynomial a Fourier transform quality. In essence, the coefficients are the product of a discrete Fourier transform of the elevation data. This will later give rise to problems of anti-aliasing when the curve fit is created.

2. Use of a Chebyshev Polynomial Curve Fit

The rationale for using a Chebyshev polynomial to fit a curve is that it will exactly fit all points considered in the fit. This type of fit uses the spectral information of the known points to determine the behavior of a function in between and over all known points. The Chebyshev polynomial is also very convenient to use since the polynomial fit at an arbitrary point need not be determined from the entire set of coefficients. There is a point in the summation where additional terms are negligible. Often times, only about 20 percent of the terms are needed to converge. In the terrain data set, it is quite easy to produce a matrix of coefficients that would represent a Chebyshev "surface." An elevation value would be simply calculated from this array of coefficients. Unfortunately, there are some problems.

Terrain data need not be continuous. Cliffs, overhangs, river gorges, canyons, mesas, or essentially any rapid transition in elevation along an isolattitude or isolongitude would present a problem in the Chebyshev fit. Remember that the Chebyshev coefficients are very similar to the coefficients generated by a Fourier transform. Consider a set of discrete points with somewhat abrupt changes in value, such as depicted in figure 1. Straight lines are drawn between the points to illustrate the effect of a linear interpolation. As is easily seen, there are some large transitions, with the largest occurring between points 1 and 2. A rough estimate of the slope is somewhat steep at $\Delta y / \Delta x = -0.46$. These points are not an unreasonable expectation of hilly terrain leading to a valley. In fact, they would represent a likely cross-section of an avenue for ground troop movement or encampment.

Figure 1. Elevation data along an arbitrary isolattitude.
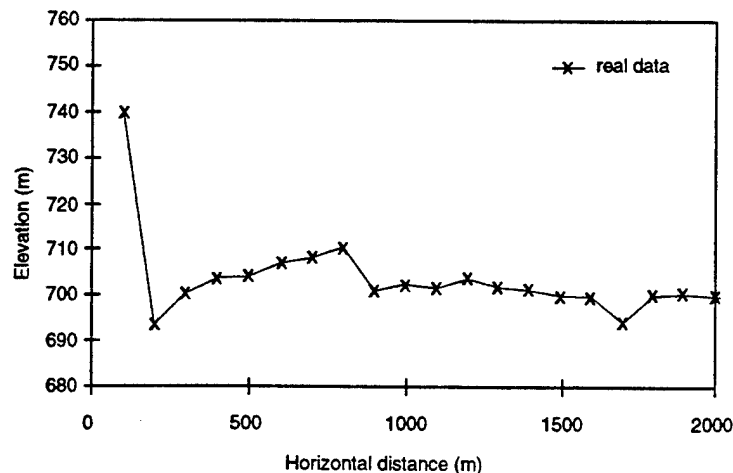
A Chebyshev polynomial fit results in a curve that is shown in figure 2. One can immediately see the effect of anti-aliasing in the first 1000 m of the terrain profile, which is brought on by the highly transient regions contributing to the value (or energy) of the high wave number terms. These terms are later reflected in the oscillations near the base of the hill as continuing, yet decaying, in amplitude. In fact, everywhere in this section of terrain, the Chebyshev fit has introduced high spectral content in the fit. This is the occurrence in the Fourier transform of a discontinuous function. This is typically not true in the termination of hills into a valley.

3. Cubic Spline Interpolation

So the question becomes, is it still more reasonable to fit terrain elevation with a Chebyshev polynomial rather than use a linear interpolation, or is there something better? Perhaps the use of a cubic spline evaluated from a few surrounding points would be a better way to account for local trends, still gain a nonlinear estimate of the terrain elevation, and yet not have effects from terrain that is quite distant from the area of interest. The cubic spline presents a compromise between the linear interpolation and a Chebyshev polynomial fit. The changes between points are represented by a rounded transition to better represent hills and bluffs. Although precipitous drops are still smoothed and remain underrepresented in the terrain fit, the results are probably better than a simple linear interpolation. Looking at the same terrain data, but using a cubic spline to fit, we see in figure 3 that these two criteria are met; namely, only local effects are considered, and the curvature (in this case, based on the calculated second derivative) is used to estimate the deviation from linearity.

A cubic spline interpolation is probably better for estimating elevations, especially in the case of multiple interpolations between actual data points (desired resolution greater than actual grid spacing). A cubic spline would resolve a smooth transition about the data point. If elevation across several points is needed (desired resolution less than actual grid spacing), linear interpolation is most likely adequate, since fine terrain features are filtered out by the larger grid spacing, and average values representing the pre-

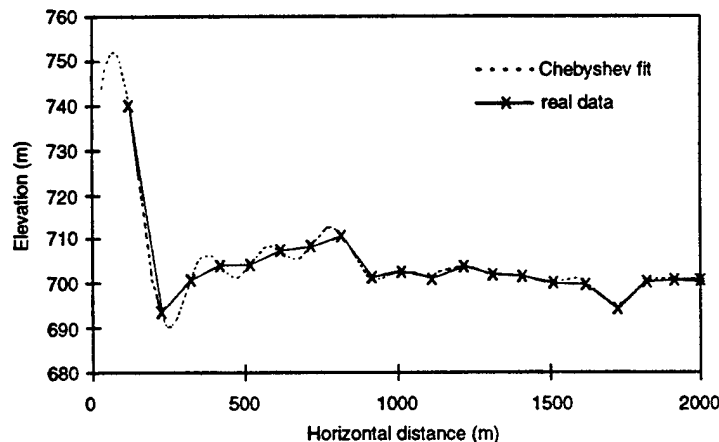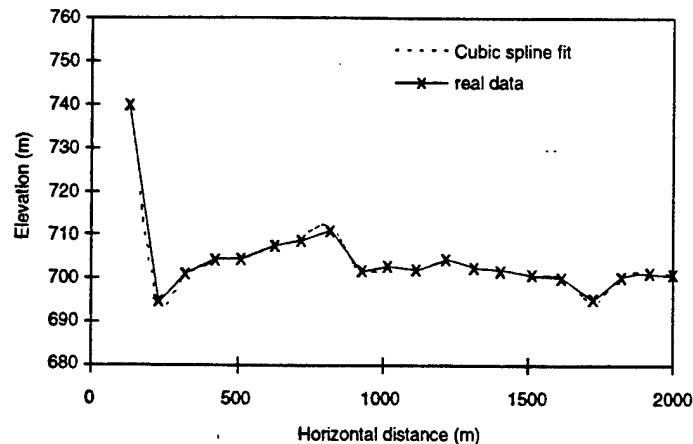**Figure 2. Chebyshev polynomial fit to terrain data.**

Figure 3. Cubic spline fit to terrain data.

dominant terrain of the area are more reasonable for a fluid dynamic computation boundary. The computational cost of evaluating the Chebyshev polynomial surface, and then extracting the interpolated values, is much more costly than performing a cubic spline interpolation, as the cubic spline interpolation is more costly than linear interpolation.

4. Conclusion

By this assessment, the cubic spline interpolation is beneficial only if grid spacing less than twice the elevation data grid spacing is needed. Error associated with linear interpolation is probably of the same order as the error associated with a cubic spline for larger grid spacing. A Chebyshev polynomial fit near a high relief area would result in excessive elevation oscillations that may not be realistic. It is recommended that a cubic spline interpolation process be used for fine grid interpolation and that linear interpolation be used for coarse grid interpolation.

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REPORT DOCUMENTATION PAGE

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	December 1996	Interim, from 1 to 31 Oct. 1996

4. TITLE AND SUBTITLE	5. FUNDING NUMBERS
Chebyshev Polynomial Fit for Terrain Elevation	DA PR: B53A PE: P61120

6. AUTHOR(S)

Richard B. Loucks

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
U.S. Army Research Laboratory Attn: AMSRL-IS-EE 2800 Powder Mill Road Adelphi, MD 20783-1197	ARL-TN-86

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSORING/MONITORING AGENCY REPORT NUMBER
U.S. Army Research Laboratory 2800 Powder Mill Road Adelphi, MD 20783-1197	

11. SUPPLEMENTARY NOTES

AMS code: 611102.53A11
ARL PR: 7FEJ60

12a. DISTRIBUTION/AVAILABILITY STATEMENT	12b. DISTRIBUTION CODE
Approved for public release; distribution unlimited.	

13. ABSTRACT (Maximum 200 words)

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14. SUBJECT TERMS			15. NUMBER OF PAGES
Chebyshev, terrain, fit			13
			16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	UL

NSN 7540-01-280-5500

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